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Korovkin Theorems for a Class of Integral Operators

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INTRODUCTION

Recently, V. A. Baskakov [1] introduced a class of linear operators on $C[a, b]$ that is more general than positive operators, and obtained various convergence theorems of Korovkin type. The theorems in [1] are of three types: (1) convergence of a bounded sequence of operators from his class to the identity on a certain test set implies the convergence for all f in $C[a, b]$; (2) convergence of a sequence of operators from his class to the lth-derivative on the test set implies the same sort of convergence for all $f \in C^k[a, b]$; and (3) theorems limiting the degree of convergence of sequences of polynomial valued operators belonging to his class.

In the present paper, we shall combine the idea of Baskakov with the concept of finite oscillation kernels to obtain a wider class of operators and the corresponding theorems. To this end, we shall refer to many of the results by M. J. Marsden and the author [4]; particularly, those results dealing with disconjugate differential equations.

Let $L_m y = D^m y + \sum_{j=1}^{m} a_j(t) D^{j-1} y = 0$, $D = d/dt$, be a linear differential equation with continuous coefficients, i.e., $a_i \in C(\alpha, \beta)$. We suppose that any solution of $L_m y = 0$ has $m - 1$ or fewer zeros in $[\alpha, \beta]$, i.e., $L_m y$ is disconjugate on $[\alpha, \beta]$. In such a case, there are functions $\xi_i \in C^{m+1-i}(\alpha, \beta)$, $\xi_i > 0$ on (α, β) such that L_m can be factored as

$$L_m \equiv \omega(t) D_m \cdot \cdots \cdot D_2 D_1 \tag{1.1}$$

where $D_i y = D(y/\xi_i)$ and $\omega = 1/\xi_{m+1} = \xi_1 \cdot \cdots \cdot \xi_m$. The functions $\xi_2, \ldots, \xi_m$ are only integrable on proper subintervals $[\alpha, c]$ of $[\alpha, \beta]$, and the set of functions $u_1, \ldots, u_m$ defined by $D_j D_{j-1} \ldots D_1 u_{j+1} = \xi_{j+1}$, $D^k u_{j+1}(\alpha) = 0$ for $k = 0, \ldots, j - 1$, is called a fundamental principal system for L_m on $[\alpha, \beta]$.

Moreover, a fundamental principal system on $[\alpha, \beta]$ is unique up to multiplication by positive constants. In the discussion that follows, we adopt the notation,

$$\mathscr{D}^l y = D_l D_{l-1} \cdot \cdots \cdot D_1 y \qquad l \leqslant m, \tag{1.2}$$

where (1.2) is obtained from the decomposition in (1.1). For the above facts and a more complete discussion of disconjugate equations see Willett [5] and the references therein.

Let $L_k y = 0$ be a disconjugate equation on $[\alpha, \beta]$ with $[a, b] \subset (\alpha, \beta)$. We define a class of linear operators $\mathscr{S}_m(L_k : [\alpha, \beta])$ on the space X to itself by:

(i) $A \in \mathscr{S}_m(L_k : [\alpha, \beta])$ implies $A[f; x] = \int_a^b f(t)\, K(x, t)\, dt$

(ii) for each $x \in (a, b)$, and for the function

$$S_x(t) = \begin{cases} \displaystyle\int_a^t \xi_k(t_k) \int_a^{t_k} \xi_{k-1}(t_{k-1}) \cdots \int_a^{t_2} \xi_1(t_1)\, K(x, t_1)\, dt_1 \cdots dt_k\,, & a < t < x, \\ \displaystyle\int_t^b \xi_k(t_k) \int_{t_k}^b \xi_{k-1}(t_{k-1}) \cdots \int_{t_2}^b \xi_1(t_1)\, K(x, t_1)\, dt_1 \cdots dt_k\,, & x < t < b, \end{cases} \tag{1.3}$$

there is a partition of $[a, b]$ into at most $m + 1$ intervals $I_{1,x}, \ldots, I_{r+1,x}$, $(r = r(x))$, such that $S_x(t)(x - t)^k$ is of one sign ($\geqslant 0$ or $\leqslant 0$) on each $I_{j,x}$, and $S_x(t)(x - t)^k$ alternates in sign on these intervals. The functions ξ_j, $j = 1, \ldots, k$ in Eq. (1.3) correspond to the decomposition of L_k on $[\alpha, \beta]$ as described above.

The space X in the above definition may be taken as $C[a, b]$, or, as a Banach space of Lebesgue measurable functions on $[a, b]$ which satisfies (a) $C^{m+k}[a, b] \subset X$ and is dense, (b) $X \subseteq L^1[a, b]$, i.e., $\|f\|_{L_1} \leqslant M \|f\|_X$, $f \in X$ and M an absolute constant, and (c) $|g| \leqslant |f|, f \in X$ implies $g \in X$ and $\|g\|_X \leqslant C \|f\|_X$ where C is an absolute constant. In the first case, $K(x, t)\, dt = d\alpha_x(t)$ where $\alpha_x(t)$ is of bounded variation on $[a, b]$, and in the latter case, $K(x, t)$ is an $[a, b] \times [a, b]$ Lebesgue measurable function and condition (ii) is satisfied for almost all x.

The class of V. A. Baskakov [1] is covered by taking $m = 0$, k even, and L^k to be ordinary differentiation k-times which is disconjugate on the interval $[0, \infty)$ (i.e., the weight functions $\xi_i \equiv 1$).

In the sequel, if α is not a singular point for the disconjugate equations involved, then $a = \alpha$ may be included in the definition and theorems. Further, we shall assume $k \geqslant 1$, since the case $k = 0$ corresponds to the class $\mathscr{S}_m$ of [4].

2. The Main Theorems

Suppose that $L_{m+1}y = 0$ is a disconjugate equation on $[\alpha, \beta]$, $[a, b] \subset (\alpha, \beta)$, with a fundamental principal system on $[\alpha, \beta]$ given by $(v_1, ..., v_{m+1})$. Let $L_k y = 0$ be the disconjugate equation with fundamental principal system $(u_1, ..., u_k)$ on $[\alpha, \beta]$ which defines the class $\mathscr{S}_m(L_k : [\alpha, \beta])$. For $j = 1, ..., m+1$, we define u_{k+j} as

$$u_{k+j}(t) = \xi_1(t) \int_\alpha^t \xi_2(\tau_1) \int_\alpha^{\tau_1} \cdots \int_\alpha^{\tau_{k-2}} \xi_k(\tau_{k-1}) \int_\alpha^{\tau_{k-1}} v_j(\tau_k)\, d\tau_k \cdots d\tau_1, \qquad (2.1)$$

where the ξ_i are as in (1.3). Clearly, $\mathscr{D}^k u_j = v_j$.

Theorem 1. *Suppose that* $\{A_n\} \subset \mathscr{S}_m(L_k : [\alpha, \beta])$ *on* X. *Let* $L_{m+1}y = 0$ *be disconjugate on* $[\alpha, \beta]$, *and suppose* $(u_1, ..., u_k, u_{k+1}, ..., u_{k+m+1})$ *are as above. Then, the conditions* $\| A_n \|_X \leqslant M < +\infty$, *and* $\| A_n[u_i : x] - u_i(x)\|_X \to 0$ *as* $n \to \infty$ *for* $i = 1, 2, ..., m+k+1$, *imply* $\| A_n[f: x] - f(x)\|_X \to 0$ *as* $n \to \infty$ *for each* $f \in X$.

If the operators L_{m+1} and L_k correspond to ordinary differentiation (the weights ξ_i in (1.3) and (2.1) taken to be identically 1), then we obtain the following.

Corollary 1. *If* $\{A_n\} \subset \mathscr{S}_m(D^k: [0, \infty))$ *on* X, *then* $\| A_n \|_X \leqslant M < +\infty$ *and* $\| A_n[t^j: x] - x^j \|_X \to 0$ *as* $n \to \infty$ *for* $j = 0, 1, ..., m+k$, *imply* $\| A_n[f: x] - f(x)\|_X \to 0$ *for all* $f \in X$.

For the class $\mathscr{S}_m(L_k : [\alpha, \beta])$, we can give an analogous theorem concerning convergence to certain "generalized" derivatives given by Eq. (1.2).

Theorem 2. *Let* $\{A_n\} \subset \mathscr{S}_m(L_k : [\alpha, \beta])$ *on* X, *and* $(u_1, ..., u_k, u_{k+1}, ..., u_{k+m+1})$ *be as above. Then the convergence*

$$\| A_n[u_i : x] - \mathscr{D}^l u_i(x)\|_X \to 0, \qquad i = 1, ..., m+k+1,$$

$l < k$, *implies* $\| A_n[f: x] - \mathscr{D}^l f(x)\|_X \to 0$ *for all* $f \in X$ *with* $\mathscr{D}^{k+m} f \in C[a, b]$.

In the case of polynomial valued operators of class $\mathscr{S}_m(L_k : [\alpha, \beta])$ on X, we can give an estimate on the degree of convergence. Let $\mathbf{P}_n$ denote the class of polynomials of degree not exceeding n.

Theorem 3. *Suppose that* (i) $\{A_n\} \subset \mathscr{S}_m(L_k : [\alpha, \beta])$ *on* X, (ii) $A_n f \in \mathbf{P}_n$ *for each* $f \in X$, *and* (iii) L_{m+1} *and* $(u_1, ..., u_k, u_{k+1}, ..., u_{k+m+1})$ *are as in Theorem* 1. *Then at least one of the sequences*

$$n^{m+k} \| A_n[u_j : x] - u_j(x)\|_X, \qquad j = 1, ..., m+k+1,$$

does not converge to zero.

As in [4], a more careful look at the clustering nature of the sign change points of $S_x(t)(x-t)^k$ yields a stronger result. Let $t_{1,x,n},..., t_{r,x,n}$ be the endpoints of the intervals $I_{1,x,n},..., I_{r+1,x,n}$ contained in the interior of $[a, b]$. Let j_0 represent the *essential number* of *sign changes* for the sequence $\{A_n\}$, i.e., j_0 is the smallest number j_0 for which there exists $\delta > 0$ and n_0 such that for each x (or x a.e.), and each $n \geqslant n_0$, at most j_0 of the points $t_{i,x,n}$ lie in any interval of length δ.

THEOREM 4. *If, in addition to the assumptions of Theorem* 3, $\{A_n\}$ *has* j_0 *essential sign changes, then at least one of the sequences*

$$n^{j_0+k} \| A_n[u_j : x] - u_j(x)\|_X, \qquad j = 1,..., m + k + 1,$$

does not converge to zero.

A quantitative result corresponding to the convergence in Theorem 2 can also be obtained.

THEOREM 5. *Suppose that* (i) $\{A_n\} \subset \mathscr{S}_m(L_k : [\alpha, \beta])$ on X, (ii) $A_n f \in \mathbf{P}_n$ *for each* $f \in X$, and (iii) L_{m+1} *and* $(u_1,..., u_k, u_{k+1},..., u_{k+m+1})$ *are as above. If* A_n *has* j_0 *essential sign changes, then at least one of the sequences*

$$n^{j_0+k-l} \| A_n[u_j : x] - \mathscr{D}^l u_j(x)\|_X, \qquad j = 1,..., m + k + 1,$$

$l < k$, *does not converge to zero.*

Finally, as a corollary to the proof of Theorem 1, we can obtain a quantitative statement in the other direction for a sequence of operators $\{A_n\} \subset \mathscr{S}_m(L_k : [\alpha, \beta])$ on $C[a, b]$.

THEOREM 6. *Let* $f \in C[a, b]$, $\{A_n\} \subset \mathscr{S}_m(L_k : [\alpha, \beta])$ *on* $C[a, b]$, $\| A_n \|_{C[a,b]} \leqslant M_1 < +\infty$, *and* L_{m+1} *and* $(u_1,..., u_k, u_{k+1},..., u_{k+m+1})$ *as in Theorem* 1. *If*

$$\| A_n[u_i : x] - u_i(x)\|_{C[a,b]} \leqslant M_2 \sigma_n^{m+k},$$

$i = 1,..., k + m + 1$, $\sigma_n \to 0$, *then*

$$\| A_n[f : x] - f(x)\|_{C[a,b]} \leqslant M_3 \sigma_n^{m+k} + M_4 \omega_{m+k}(f, \sigma_n),$$

where $\omega_{m+k}(f, \sigma_n)$ *is the* $(m + k)$*th modulus of smoothness.*

3. The Interpolating Polynomials

Let $f \in C^{m+k}[a, b]$. As was established in [4], we can interpolate such a function by "polynomials" comprised of elements from a fundamental principal system. In this section, we shall set up the interpolating system and obtain boundedness of its coefficients.

Let $t_i = t_{i,x,n}$, $i = 1,..., r$, $r \leq m$, be the endpoints of the intervals $I_{1,x,n},..., I_{r+1,x,n}$ interior to $[a, b]$. We interpolate $\mathscr{D}^k f$ at the points $t_1,..., t_r$ using the fundamental principal system $(v_1,..., v_{m+1})$ of the disconjugate equation $L_{m+1}y = 0$ on $[\alpha, \beta]$. Indeed, let

$$J(t) = \begin{pmatrix} v_1 & \cdots & v_r & \mathscr{D}^k f \\ t_1 & \cdots & t_r & t \end{pmatrix} \Big/ \begin{pmatrix} v_1 & \cdots & v_r \\ t_1 & \cdots & t_r \end{pmatrix} = (C)/(D). \tag{3.1}$$

(see [4, Eq. 3.6]) where the brackets represents the determinant of the matrix whose (i, j)-entry is the ith function evaluated at the jth point.

Note that $J(t)$ may be written in the form

$$J(t) = \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right) \Big/ \left(\begin{array}{c|c} A & 0 \\ \hline B & D \end{array}\right), \tag{3.2}$$

where

$$A = \begin{bmatrix} \xi_1(x) & 0 & & \mathbf{0} \\ \xi_2(x) & \xi_2(x) & & \\ \vdots & & \ddots & \\ u_k(x) & \mathscr{D}^1 u_k(x) & \cdots & \xi_k(x) \end{bmatrix}, \qquad B = \left[\frac{B'}{f(x)\ \mathscr{D}^1 f(x) \cdots \mathscr{D}^{k-1} f(x)}\right]$$

and $B' = [\mathscr{D}^{j-1} u_{k+i}(x)]$ $(j = 1,..., k,\ i = 1,..., r)$.

Integrating $J(t)$ against the weights ξ_i, we obtain

$$\begin{aligned} R(t) &= \xi_1(t) \int_x^t \xi_2(\tau_1) \int_x^{\tau_1} \cdots \xi_k(\tau_{k-1}) \int_x^{\tau_{k-1}} J(\tau_k)\, d\tau_k \cdots d\tau_1 \\ &= \left(\begin{array}{c|c|c} A & 0 & \\ \hline B & C' & U_f(t)^T \end{array}\right) \Big/ \left(\begin{array}{c|c} A & 0 \\ \hline B' & D \end{array}\right), \end{aligned} \tag{3.3}$$

where $C = [C' : V_f^T]$, $U_f(t) = [u_1(t) \ldots u_k(t)\ u_{k+1}(t) \ldots u_{k+r}(t)\ f(t)]$ and $V_f = [v_1(t) \ldots v_r(t)\ \mathscr{D}^k f(t)]$.

Observe that $R(t) = f(t) - p_{x,n}(t)$, where

$$p_{x,n}(t) = \sum_{j=1}^{k+m+1} a_j(x, n)\, u_j(t) \tag{3.4}$$

and that $p_{x,n}(t)$ interpolates f and its generalized derivatives of order up to $k-1$ at x, i.e., $\mathscr{D}^j p_{x,n}(x) = \mathscr{D}^j f(x)$, $j = 0, 1, \ldots, k-1$.

We shall need the coefficients $a_j(x, n)$ to be uniformly bounded. Now,

$$| a_j(x, n)| = \left| \left(\begin{array}{c|c} A & 0 \\ \hline B & C' \end{array}\right)_j \Big/ \left(\begin{array}{c|c} A & 0 \\ \hline B' & D \end{array}\right) \right|,$$

where $(\ )_j$ denotes that the jth row has been removed. Using the triangular nature of A and an inductive procedure, we can write $a_j(x, n)$ as a linear combination of determinants of the form

$$\left(\begin{array}{c|c} \mathscr{D}^i U_f(x)^T & C' \end{array}\right) \Big/ \left(\begin{array}{c|c} A & 0 \\ \hline B' & D \end{array}\right)$$

where $\mathscr{D}^i U_f(x) = [\mathscr{D}^i u_{k+1}(x) \ldots \mathscr{D}^i u_{k+r}(x)\ \mathscr{D}^i f(x)]$, $0 \leqslant i \leqslant k-1$, which in turn are linear combinations of

$$\begin{pmatrix} v_1 & \cdots & v_{j-1} & \mathscr{D}^k f & v_{j+1} & \cdots & v_r \\ t_1 & \cdots & t_{j-1} & t_j & t_{j+1} & \cdots & t_r \end{pmatrix} \Big/ \begin{pmatrix} v_1 & \cdots & v_r \\ t_1 & \cdots & t_r \end{pmatrix}. \tag{3.5}$$

The last determinants are uniformly bounded by Lemma 3.2 of [4]. The coefficients in the above linear combinations involve products (in various combinations) of the function $\xi_i(x)$, $1/\xi_j(x)$, $\mathscr{D}^i f(x)$, and $\mathscr{D}^i u_j(x)$, all of which are uniformly bounded on the interval $[a, b]$.

The function $J(t)$ also can be written as

$$\begin{aligned} J(t) &= \left[\begin{pmatrix} v_1 & \cdots & v_r & \mathscr{D}^k f \\ t_1 & \cdots & t_r & t \end{pmatrix} \Big/ \begin{pmatrix} v_1 & \cdots & v_r & v_{r+1} \\ t_1 & \cdots & t_r & t \end{pmatrix}\right] \left[\begin{pmatrix} v_1 & \cdots & v_r & v_{r+1} \\ t_1 & \cdots & t_r & t \end{pmatrix} \Big/ \begin{pmatrix} v_1 & \cdots & v_r \\ t_1 & \cdots & t_r \end{pmatrix}\right] \\ &= g_{x,n}(t)\, q_{x,n}(t). \end{aligned}$$

According to Lemma 3.2 of [4], $| g_{x,n}(t)|$ is bounded uniformly in t, x, n. Further, we note that

$$q_{x,n}(t) = \left(\begin{array}{c|c} A & 0 \\ \hline B_1 & C_1 \end{array}\right) \Big/ \left(\begin{array}{c|c} A & 0 \\ \hline B' & D \end{array}\right)$$

where B_1 and C_1 are found from B and C, respectively, by replacing f by u_{k+r+1}. Thus,

$$R_1(t) = \xi_1(t) \int_x^t \xi_2(\tau_1) \int_x^{\tau_1} \cdots \xi_k(\tau_{k-1}) \int_x^{\tau_{k-1}} q_{x,n}(\tau_k)\, d\tau_k \cdots d\tau_1, \tag{3.6}$$

is found from R by replacing f by u_{k+r+1}. Thus, $R_1(x) = 0$ and

$$R_1(t) = \sum_{j=1}^{k+m+1} b_j(x, n)\, u_j(t), \tag{3.7}$$

where the $b_j(x, n)$ are uniformly bounded.

Finally, we note that $q_{x,n}(t)$ or $-q_{x,n}(t)$ agrees in sign with $S_x(t)(x - t)^k$.

4. Proofs of Theorems 1 and 2

4.1 *Proof of Theorem* 1. By the uniform boundedness principle, it suffices to show the convergence for $f \in C^{m+k}[a, b]$. Let $f \in C^{m+k}[a, b]$ and x be given. By (3.4), we have

$$A_n[f : x] - f(x) = \sum_{j=1}^{k+m+1} a_j(x, n)\{A_n[u_j : x] - u_j(x)\} + A_n[R(\cdot) : x], \tag{4.1}$$

where the coefficients $a_j(x, n)$ are uniformly bounded.

In order to estimate $A_n[R(\cdot): x]$, we decompose the integral over $[a, x]$ and $[x, b]$, and apply (3.3) to obtain

$$\begin{aligned} &A_n[R(\cdot) : x] \\ &\quad = \int_a^x \xi_1(t)\, K_n(x, t)(-1)^k \int_t^x \xi_2(\tau_1) \int_{\tau_1}^x \cdots \xi_k(\tau_{k-1}) \int_{\tau_{k-1}}^x J(\tau_k)\, d\tau_k \cdots d\tau_1\, dt \\ &\qquad + \int_x^b \xi_1(t)\, K_n(x, t) \int_x^t \xi_2(\tau_1) \int_x^{\tau_1} \cdots \xi_k(\tau_{k-1}) \int_x^{\tau_{k-1}} J(\tau_k)\, d\tau_k \cdots d\tau_1\, dt. \end{aligned}$$

Interchanging the order of integration yields

$$\begin{aligned} A_n[R(\cdot) : x] &= \int_a^x J(t)\, S_x(t)(-1)^k\, dt + \int_x^b J(t)\, S_x(t)\, dt \\ &= \int_a^x g_{x,n}(t)\, q_{x,n}(t)\, S_x(t)(-1)^k\, dt + \int_x^b g_{x,n}(t)\, q_{x,n}(t)\, S_x(t)\, dt. \end{aligned}$$

Since $q_{x,n}(t)$ changes sign with $S_x(t)(x - t)^k$ and $|\, g_{x,n}(t)| \leqslant M < +\infty$ uniformly, we have

$$|\, A[R(\cdot) : x]| \leqslant M \left| \int_a^x q_{x,n}(t)\, S_x(t)(-1)^k\, dt + \int_x^b q_{x,n}(t)\, S_x(t)\, dt \right|.$$

From (3.6) and (3.7) and interchanging integration, it follows that

$$|A[R(\cdot):x]| \leqslant M \,|\, A_n[R_1(\cdot):x]| \leqslant M \sum_{j=1}^{k+m+1} |\, b_j(x,n)|\,|\, A_n[u_j:x] - u_j(x)| \,. \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\|\, A_n[f:x] - f(x)\|_X \leqslant M' \sum_{j=1}^{k+m+1} \|\, A_n[u_j:x] - u_j(x)\|_X \,, \tag{4.3}$$

where M' depends only on the generalized derivatives of f and the u_j. Theorem 1 follows immediately.

4.2 *Proof of Theorem* 2. We can interpolate $\mathscr{D}^j f$, $j = 0, 1,\ldots, l-1$ at x by means of the system $(u_1,\ldots, u_k)$. Indeed, $f(t) = p_{x,n}(t) + R_2(t)$ where

$$R_2(t) = \begin{pmatrix} u_1 & \cdots & u_l & f \\ x & \cdots & x & t \end{pmatrix} \Big/ \begin{pmatrix} u_1 & \cdots & u_l \\ x & \cdots & x \end{pmatrix},$$

defines the interpolating polynomials. As was shown in [4], the coefficients of $p_{x,n}(t) = \sum_{j=1}^{l} a_j(x,n)\, u_j(t)$ are uniformly bounded in x and n (the bound depends on the system $(u_1,\ldots, u_l)$ and the derivatives to order $l-1$ of f).

Thus,

$$\begin{aligned} &\|\, A_n[f:x] - \mathscr{D}^l f(x)\|_X \\ &\quad\leqslant M \sum_{j=1}^{l} \|\, A_n[u_j:x]\|_X + \|\, A_n[R_2(\cdot):x] - \mathscr{D}^l f(x)\|_X \\ &\quad= M \sum_{j=1}^{l} \|\, A_n[u_j:x] - \mathscr{D}^l u_j(x)\|_X + \|\, A_n[R_2(\cdot):x] - \mathscr{D}^l f(x)\|_X \,. \end{aligned}$$

Consequently, we only need to estimate this last term.

The form of R_2 is quite simple since all the interpolation takes place at x. Indeed,

$$R_2(t) = \frac{1}{\xi_1(x)\cdots\xi_l(x)} \begin{pmatrix} \xi_1(x) & & & 0 & u_1(t) \\ u_2(x) & \xi_2(x) & & & u_2(t) \\ \vdots & & \ddots & & \vdots \\ u_l(x) & D_1 u_l(x) & \cdots & \xi_l(x) & u_l(x) \\ f(x) & D_1 f(x) & \cdots & \mathscr{D}^{l-1} f(x) & f(t) \end{pmatrix}.$$

From this, it follows that

$$R_2(t) = \xi_1(t) \int_x^t \xi_2(\tau_1) \int_x^{\tau_1} \cdots \xi_l(\tau_{l-1}) \int_x^{\tau_{l-1}} \mathscr{D}^l f(\tau_l)\, d\tau_l \cdots d\tau_1 \,.$$

Consequently, by decomposing the integration over $[a, x]$ and $[x, b]$, and interchanging the order of integration, we have

$$\begin{aligned} A_n[R_2 : x] &= \int_a^x \mathscr{D}^l f(\tau_l)(-1)^l \int_{\tau_l}^x \xi_l(\tau_{l-1}) \cdots \int_{\tau_1}^x \xi_1(t)\, K_n(x, t)\, dt\, d\tau_1 \cdots d\tau_l \\ &\quad + \int_x^b \mathscr{D}^l f(\tau_l) \int_x^{\tau_l} \xi_l(\tau_{l-1}) \cdots \int_x^{\tau_1} \xi_1(t)\, K_n(x, t)\, dt\, d\tau_1 \cdots d\tau_l \\ &= A_n{}^*[\mathscr{D}^l f : x], \end{aligned}$$

where the kernel of $A_n{}^*$ is understood from the equation.

We claim that the sequence $\{A_n{}^*\} \subset \mathscr{S}_m(L_{k-l} : [\alpha, \beta])$ where $L_{k-l}y = \bar{\omega}(t)\, D_k \ldots D_{l+1}y$ with fundamental principal system $(w_1, \ldots, w_{k-l})$ defined by $\mathscr{D}^l u_j = w_{j-l}$. The functions $\bar{\xi}_j$ corresponding to this system are just $\bar{\xi}_j = \xi_{j+l}$. Thus, the function $S_{x,k-l}(t)$ used in the definition of $\mathscr{S}_m(L_{k-l} : [\alpha, \beta])$ is $S_{x,k-l}(t) = (-1)^l S_x(t)$, $a < t < x$, and $S_{x,k-l}(t) = S_x(t)$, $x < t < b$. Observe that the number of sign changes of $S_{x,k-l}(t)(x-t)^{k-l}$ are the same as the sign changes of $S_x(t)(x-t)^{k-l}(-1)^l(x-t)^l = S_x(t)(x-t)^k(-1)^l$ Thus, $\{A_n{}^*\} \subset \mathscr{S}_m(L_{k-l} : [\alpha, \beta])$.

We now show that the proof of Theorem 1 applies to $\{A_n{}^*\}$, L_{m+1} and L_{k-l}. Let $(w_1, \ldots, w_{k-l}, w_{k-l+1}, \ldots, w_{k-l+m+1})$ be the system as in Theorem 1 corresponding to L_{m+1} and L_{k-l} i.e., w_{k-l+j} is defined as in (2.1) using the weights $\bar{\xi}_i$ corresponding to L_{k-l}. Now,

$$\begin{aligned} &A_n{}^*[w_j : x] \\ &\quad = \int_a^b \xi_1(t)\, K_n(x, t) \int_t^x \xi_2(\tau_1) \cdots \int_{\tau_{l-2}}^x \xi_l(\tau_{l-1}) \int_{\tau_{l-1}}^x w_j(\tau_l)\, d\tau_l \cdots d\tau_1\, dt \\ &\quad = \int_a^b K_n(x, t)\, R_3(t)\, dt, \end{aligned}$$

where

$$R_3(t) = \begin{pmatrix} u_1 & \cdots & u_l & u_{l+j} \\ x & \cdots & x & t \end{pmatrix} \Big/ \begin{pmatrix} u_1 & \cdots & u_l \\ x & \cdots & x \end{pmatrix},$$

since $w_j(\tau_l) = \mathscr{D}^l u_{l+j}(\tau_l)$, $j = 1, \ldots, m+1$. Notice that $R_3(t)$ defines a method of interpolating $\mathscr{D}^i u_{l+j}$, $i = 0, 1, \ldots, l-1$ at x by a polynomial in $(u_1, \ldots, u_l)$. Furthermore, the coefficients of the interpolating polynomial are uniformly bounded. Consequently,

$$\begin{aligned} \| A_n{}^*[w_j : x] - w_j(x) \|_X &= \| A_n[R_3 : x] - \mathscr{D}^l u_{l+j}(x) \|_X \\ &\leqslant \| A_n[u_{l+j} : x] - \mathscr{D}^l u_{l+j}(x) \| + M \sum_{j=1}^{l} \| A_n[u_j : x] \|_X, \end{aligned}$$

where M is independent of n. By the conditions of Theorem 2,

$$\| A_n^*[w_j : x] - w_j(x)\|_X \quad \text{tends to zero as } n \to \infty.$$

If $f \in C^{m+k}[a, b]$, then $\mathscr{D}^l f \in C^{m+k-l}[a, b]$ and the proof of Theorem 1 yields

$$\| A_n^*[\mathscr{D}^l f: x] - \mathscr{D}^l f(x)\|_X \to 0 \qquad \text{as} \quad n \to \infty.$$

This completes the proof of Theorem 2.

The following statement is an immediate consequence of the above proof.

COROLLARY 2. *If the* $\{A_n^*\}$ *defined above turn out to have uniformly bounded norms, then the convergence in Theorem 2 holds for functions* $f \in C^l[a, b] \cap X$.

5. PROOFS OF THEOREMS 3, 4, 5 AND 6

From (4.3) and well-known bounds on derivatives for $f \in C^{m+k}[a, b]$, there holds

$$\| A_n[f : x] - f(x)\|_X \leqslant M(\|f\|_{L^\infty} + \|f^{(m+k)}\|_{L^\infty}) \sum_{j=1}^{k+m+1} \| A_n[u_j : x] - u_j(x)\|_X , \tag{5.1}$$

where M is a constant independent of f and n. Precisely as in [4], there are f_n with $\|f_n\|_{L^\infty} \leqslant 1$, $\|f_n^{(m+k)}\|_{L^\infty} \leqslant 1$ such that

$$\| A_n[f_n : x] - f_n(x)\|_{L^1} \geqslant C/n^{m+k}.$$

Theorem 3 follows immediately.

For Theorem 4, we observe that the bounds in (4.3) came from estimating the coefficients of (3.4) and (3.7). Careful consideration of these estimates show that derivatives of f of order greater than k only enter through bounding (3.5). But these were precisely the determinants considered in [4].

Theorem 5 follows from Theorem 4, Eq. (4.4), and the fact that the sign changes for A_n^* correspond to those for A_n. Indeed, by Theorem 4 and the sign properties, at least one of the sequences $n^{j_0+k-l} \| A_n^*[w_j : x] - w_j(x)\|_X$ does not converge to zero. The theorem follows by comparing this to (4.4).

Theorem 6 requires a lemma of Freud and Popov [3] and follows an argument of Ditzian and Freud [2]. The result of Freud and Popov claims that for arbitrary $f \in C[a, b]$ and $0 < h < 1$, there exists $\phi_h \in C^{m+k}[a, b]$ such that

$$\|f(x) - \phi_h(x)\|_{C[a,b]} \leqslant M_4 \omega_{m+k}(f, h) \tag{5.2}$$

and

$$\| \phi_h^{(m+k)}(x)\| \leqslant M_5 h^{-m-k} \omega_{m+k}(f, h). \tag{5.3}$$

For each n, approximate f by ϕ_n corresponding to $h = \sigma_n$. From the triangular inequality

$$\begin{aligned}\| A_n[f: x] - f(x)\|_{C[a,b]} \leqslant \| A_n \| \|f(x) - \phi_n(x)\|_{C[a,b]} \\ + \| A_n[\phi_n : x] - \phi_n(x)\|_{C[a,b]} \\ + \|f(x) - \phi_n(x)\|_{C[a,b]}.\end{aligned}$$

Using (5.1) on the middle term of the right side, applying (5.2), (5.3) and the conditions of the theorem, we obtain

$$\begin{aligned}&\| A_n[f : x] - f(x)\|_{C[a,b]} \\ &\quad \leqslant \| A_n \| M_4\omega_{m+k}(f, \sigma_n) \\ &\qquad + M(\| \phi_n(x)\|_{C[a,b]} + \| \phi_n^{(m+k)}(x)\|_{C[a,b]}) \sum_{j=1}^{m+k+1} \| A_n[u_j : x] - u_j(x)\|_{C[a,b]} \\ &\qquad + M_4\omega_{m+k}(f, \sigma_n) \\ &\quad \leqslant M_3\sigma_n^{m+k} + M_4\omega_{m+k}(f, \sigma_n).\end{aligned}$$

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